Properties of the Weibull-ARA$_\infty$ virtual age model and application in maintenance policy optimization

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Context

Context

- Complex industrial systems subjected to corrective maintenances (CM, repair), carried out after a failure.
- Maintenance are imperfect. Virtual age models are employed to characterize the general wear-out of the systems.

Motivations

- Adjusting an optimal preventive maintenance policy (periodic or dynamic) is rarely addressed in the literature considering imperfect maintenance only.
- The systems are not necessarily new at the beginning of the observations.
  - Second-hand unit.
  - Previous maintenances times are not recorded, observations are missing.
  - Consistency of the observations (new maintenance policy, systems after a burn-in period)

$\longrightarrow$ To develop theoretical properties of a classical virtual age model.

$\rightarrow$ To develop inference procedures when the initial age of the system is unknown.

$\rightarrow$ To present an optimal preventive maintenance strategy.

Modelling the maintenance process

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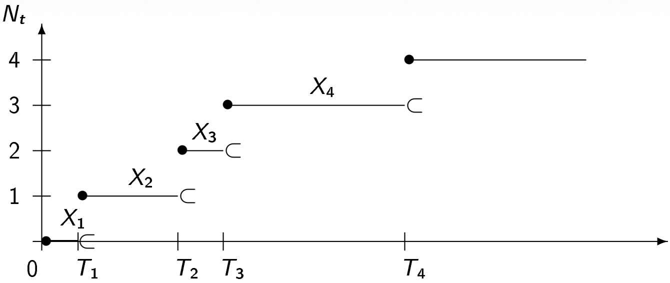


- Failure times: $\{T_i\}_{i \geq 1}$
- Inter-failure times: $X_i = T_i - T_{i-1}$, $i \geq 1$
- Counting failure process: $\{N_t\}_{t \geq 0}$, N_t = number of failures occurred at time t
- Durations of repair are not taken into account.
- Two failures cannot occur at the same time.

Stochastic modelling

$\longrightarrow$ Considering $\mathcal{H}_{t-}$ as the history of the failure process up to time t, the failure intensity λ_t is defined as:

$$\lambda_t = \lim_{dt \to 0} \frac{1}{dt} P(N_{t+dt} - N_{t-} = 1 | \mathcal{H}_{t-})$$

$\longrightarrow$ For a **self-excited point process** : $\mathcal{H}_{t-} = \sigma(\{N_s\}_{0 \leq s < t})$ and λ_t completely defines the failure process.

$\longrightarrow$ Before the first failure, the failure intensity is a deterministic and continuous function of time $\lambda(t)$, called **initial intensity** , the failure rate of T_1.

Considering industrial or software systems, a Weibull distribution is frequently used.

$$\lambda(t) = \alpha\beta t^{\beta-1} \quad , \; \alpha > 0 \; , \; \beta > 0$$

Classical models

Minimal Repair or As Bad As Old model (ABAO)

- Each maintenance leaves the system in the same state as it was before failure.
- The failure process is a Non Homogeneous Poisson Process (NHPP).

$$\lambda_t = \lambda(t)$$

Perfect repair or As Good As New model (AGAN)

- Each maintenance perfectly repairs the system and leaves it as if it were new.
- The failure process is a Renewal Process (RP).

$$\lambda_t = \lambda(t - T_{N_{t^-}})$$

Reality is between the case ABAO and AGAN

Virtual age models

After the i^{th} repair, the system performs as a new one having survived until A_i.

$$\forall i \geq 0, \quad \forall t \geq 0 P(X_{i+1} > t | X_1, ..X_i, A_i) = P(Y > A_i + t | Y > A_i) = \frac{S(A_i + t)}{S(A_i)}$$

where Y has the same distribution as X_1.

$$\lambda_t = \lambda(t - T_{N_{t^-}} + A_{N_{t^-}})$$

The A_i are called the **effective ages**. $A_0 = 0$.

- ABAO : $A_i = T_i$
- AGAN : $A_i = 0$
- The Brown-Proschan model: repairs are either perfect (AGAN) with a probability ρ, or minimal (ABAO) with a probability 1-ρ.
- The Arithmetic Reduction of Age model with memory 1 (ARA_1):

$$A_i = (1 - \rho) T_i$$

$\longrightarrow$ A virtual age model is characterized by the initial intensity and by the evolution of the effective ages.

Failure intensities of virtual age models

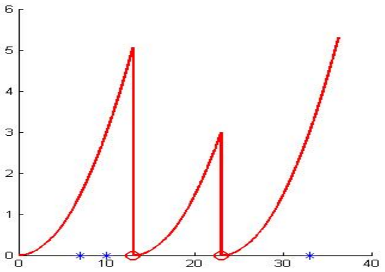


Figure: BP model and a Weibull initial intensity ($\alpha = 0.001, \beta = 3, \rho = 0.5$)

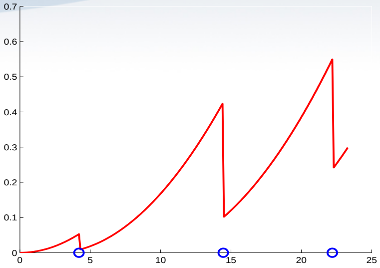


Figure: ARA_1 model and a Weibull initial intensity ($\alpha = 0.001, \beta = 3, \rho = 0.5$)

$\longrightarrow$ $\rho \in [0, 1]$ describes the maintenance efficiency.

$\longrightarrow$ Exponential distributions are not adapted as initial intensities (no ageing).

The ARA_∞ assumption

$\longrightarrow$ Arithmetic Reduction of Age model with infinite memory.

$\longrightarrow$ Brown-Mahoney-Sivazlian model (1983), particular Kijima type II model (1988), Doyen and Gaudoin (2004).

Assumption: The age of a system after maintenance is proportional to its age just before maintenance.

$$A_i = (1-\rho)(A_{i-1} + X_i)$$

$$A_i = (1-\rho)^i A_0 + \sum_{j=1}^{i} (1-\rho)^{i+1-j} X_j$$

$\longrightarrow$ Existence of a potential stationary regime of the process (Last and Szekli (1998)).

The Weibull-ARA_∞ model

Definition and simulation

$\longrightarrow$ We consider a Weibull initial intensity $\lambda(t) = \alpha\beta t^{\beta-1}$ and the ARA_∞ assumption.

$\longrightarrow$ The corresponding model is denoted Weibull-ARA_∞ model or WARA$_\infty$ model.

Simulation of effective ages under ARA_∞ assumption

$$A_{i+1} = (1-\rho)\Lambda^{-1}\left(\Lambda(A_i) + \xi_{i+1}\right)$$

where Λ is the cumulative initial intensity and Λ^{-1} its inverse, and where ξ_{i+1} is an exponential r.v. $\mathcal{E}(1)$ independent of $\{\xi_j\}_{j=1..i}$

$\Lambda(t) = \alpha t^{\beta}$ considering a Weibull initial intensity.

Effective ages for the WARA$_\infty$ model

Proposition 1: General expression of A_n

$$A_n = (1-\rho)\alpha^{-\frac{1}{\beta}} \left(\sum_{i=1}^{n} (1-\rho)^{\beta(n-i)} \xi_i \right)^{\frac{1}{\beta}}$$

with $\{\xi_j\}_{j=1..n}$ sample of exponential distribution of parameter 1.

$\longrightarrow$ Proof by induction.

$\longrightarrow$ A series of exponential distributions with different parameters follow an hypoexponential distribution.

Notation $q = (1-\rho)^\beta$

Notation q-Pochhammer series:

$$(x,x)_k = \prod_{i=1}^{k}(1-x^i) \quad , x \in \mathbb{R} \ , \quad k \in \mathbb{N}$$

Effective age distributions for the WARA$_\infty$ model

Proposition 2: Survival function of A_n

$$R_{A_n}(t) = \sum_{k=1}^{n} \frac{1}{(q,q)_{n-k}(\frac{1}{q},\frac{1}{q})_{k-1}} e^{-\frac{\alpha t^\beta}{q^k}}$$

$\longrightarrow$ " Hypo-Weibull " distribution.

$\longrightarrow$ Given the effective A_n, the distribution of the next inter-failure time can be determined.

Proposition 3: (Marginal) Survival function of X_{n+1}

$$R_{X_{n+1}}(t) = \sum_{k=1}^{n} \frac{\alpha\beta}{q^k(q,q)_{n-k}(\frac{1}{q},\frac{1}{q})_{k-1}} \int_0^\infty x^{\beta-1} e^{-\alpha(x+t)^\beta+\alpha(1-q^{-k})x^\beta}\, dx$$

Effective age distributions for the WARA$_\infty$ model

Proposition 4: Limiting survival distribution of A_n

$$R_{A_\infty}(t) = \sum_{k=1}^{\infty} \frac{1}{(q,q)_\infty(\frac{1}{q},\frac{1}{q})_{k-1}} e^{-\frac{\alpha t^\beta}{q^k}}$$

$\longrightarrow$ Explicit expression of R_{X_∞}.

Proposition 5: Expected value of the age

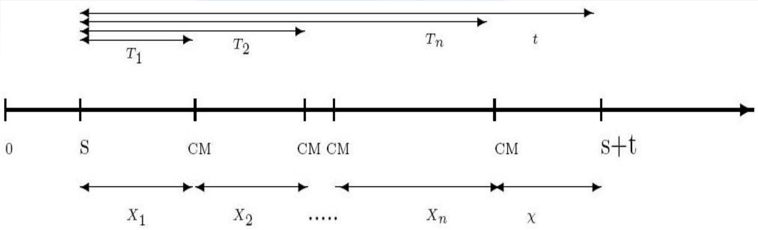
$$E[A_n] = \alpha^{-1/\beta}\Gamma(\frac{1}{\beta}+1)\sum_{k=1}^{n} \frac{q^{\frac{k}{\beta}}}{(q,q)_{n-k}(\frac{1}{q},\frac{1}{q})_{k-1}}$$

$\longrightarrow$ Derive $E[X_{n+1}], E[A_\infty], E[X_\infty]$

$\longrightarrow$ Derive the distribution of the age of the system just before the failure and its expected value in the transient and steady regime: $A_n^-, E[A_n^-], A_\infty^-, E[A_\infty^-]$.

Inference on the WARA$_\infty$ model on an observation window

The actual observations



$\longrightarrow$ The process is recorded on an obervation window $[s, s+t]$ and no information on the failure process is available prior to s.

$\longrightarrow$ No model associated to imperfect maintenance on an observation window has been developed.

$\longrightarrow$ Under the ARA$_\infty$ assumption, the only necessary information to derive the likelihood function is the initial virtual age a_0.

$$\mathcal{L}_{s,t,a_0}(x_1,\dots,x_n) = \prod_{i=1}^{n} \lambda(a_{i-1}+x_i) \times exp(-\sum_{i=1}^{n+1}[\Lambda(a_{i-1}+x_i) - \Lambda(a_{i-1})])$$

with $a_i = (1-\rho)^i a_0 + \sum_{j=1}^{i}(1-\rho)^{i+1-j} x_j$

Choice of initial age

$\longrightarrow$ In the literature, the initial age a_0 is assumed to be 0 (new system).

$\longrightarrow$ This assumption is relatively valid for a large dataset (renewal aspects of the WARA$_\infty$ model).

$\longrightarrow$ For a small dataset, the actual ageing of the system should be taken into account.

Steady regime assumption

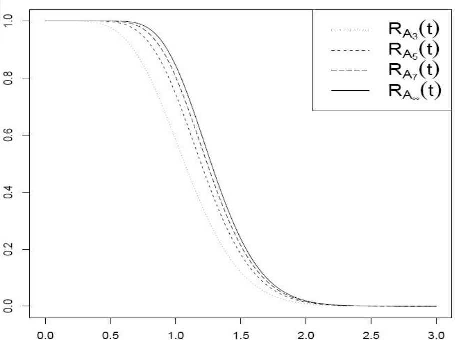


Figure: Survival functions of A_n with $(\alpha = 1, \beta = 2, \rho = 0.2)$

$\longrightarrow$ If it is likely that few maintenances have occurred, it is realistic to assume that the system is already in its stationary state.

$\longrightarrow$ The first effective age is assumed to follow the distribution of A_∞.

Likelihood function

Proposition 6: Likelihood function under steady regime

$$L^{\infty}(t_1, t_2, ..., t_n, t) =$$

$$= -\int_{(1-\rho)t_1}^{\infty} \prod_{i=1}^{n} \lambda(a_{i-1} + x_i) \exp\left(-\sum_{i=1}^{n+1} \Lambda(a_{i-1} + x_i) - \Lambda(a_{i-1}) \right) dR_{A_\infty}(h)$$

with $a_i = (1-\rho)^i h + \sum_{j=1}^{i} (1-\rho)^{i+1-j} x_j$.

$\longrightarrow$ No explicit expression of the ML estimators.

Simulations

Configuration (Example)

- $\alpha = 1$, $\beta = 4.5$, $\rho = 0.2$, $n \in \{10, 20, 30, 50\}$.
- The failure times $(t_1, ...t_n) \in [s, s+t]$ are generated so that there are an average of 100 failures before s.

Objective: Assess the surplus value of the new model

- First model (Simulated): Assume that the system is in steady regime.
- Second model: Assume that the system is As Good As New at the beginning of the observations.

$\longrightarrow$ Comparison criteria: Bias and MSE.

Results

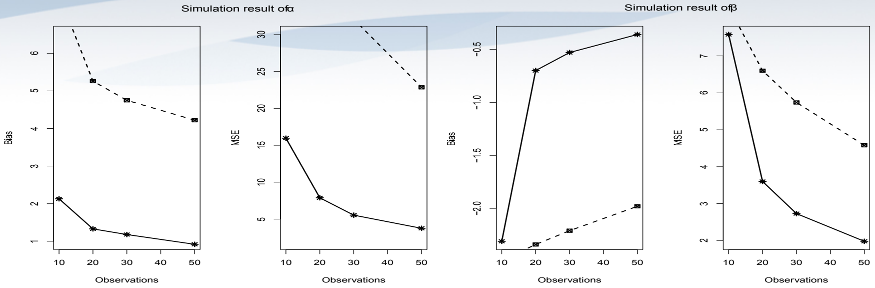
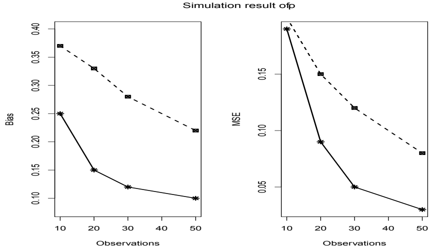


Figure: Bias and MSE with $A_0 \sim A_\infty$ (plain) and $A_0 = 0$ (dashed) of α, β and ρ



Analysis

- As the number of observations increases, the empirical bias and MSE decrease to 0.
- The estimation from the first model is always more efficient than the second model ($A_0 = 0$).
- For small ρ, the added value of the new modelling is significant.
- As ρ tends to 1, results in terms of MSE and Bias become similar for both models.

Optimal preventive maintenance strategy

Context

A repairable system is observed with the following assumptions on the history of the process:

- The ageing and the maintenance efficiency are consistent with the WARA$_\infty$ model.
- Sufficient maintenances have occurred in the past of the process so that the system is assumed to be in its stationary regime.
- Parameters of the model are assumed to be known.

$\longrightarrow$ A planned preventive maintenance (PM) policy is established on the system for maintenance cost reduction.

Assumptions

- After a maintenance (corrective or preventive) has restored the system, a new PM is scheduled after a duration **a**.
- If a failure is observed before a duration **a**, a corrective maintenance is carried out with cost C_c.
- Otherwise a PM is carried out with cost C_p ($C_p < C_c$).
- The PM efficiency and the CM efficiency are identical: ARA$_\infty$ with same parameter ρ.

$\longrightarrow$ Multiple strategies on the choice of the "age-based" duration **a** have been investigated.

First policy: Age-dependent policy

$\longrightarrow$ The duration **a** is constant during the whole process.

$\longrightarrow$ The renewal aspects of the WARA$_\infty$ model ensure that this strategy makes sense and that the long-run average cost per unit of time for an infinite horizon is finite.

$$C(a) = \lim_{t\to\infty} \frac{C_a(t)}{t} < \infty$$

$\longrightarrow$ Two choices for **a** have been studied:

- The optimal age $\mathbf{a}^*$ minimizing the cost function $C(a)$ is obtained by Monte Carlo simulations.
- As the process has similarities with a Renewal Process with generic distribution X_∞, it is possible to approach the optimal solution by optimizing the classical age-based maintenance strategy and to obtain an age $\widehat{\mathbf{a}}$.

$$\widehat{a} = arg\min_a \frac{C_p + (C_c - C_p)(1 - R_{X_\infty}(a))}{\int_0^a R_{X_\infty}(u)du} \qquad (1)$$

Evolution of the cost for a static policy

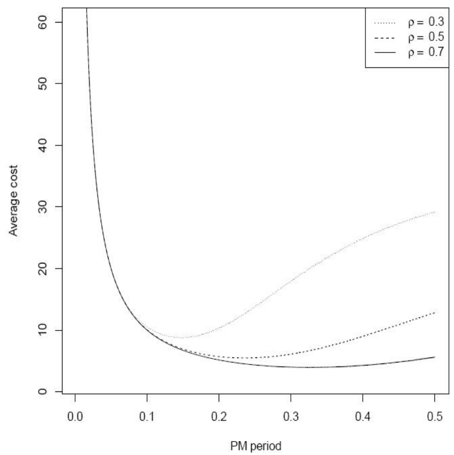
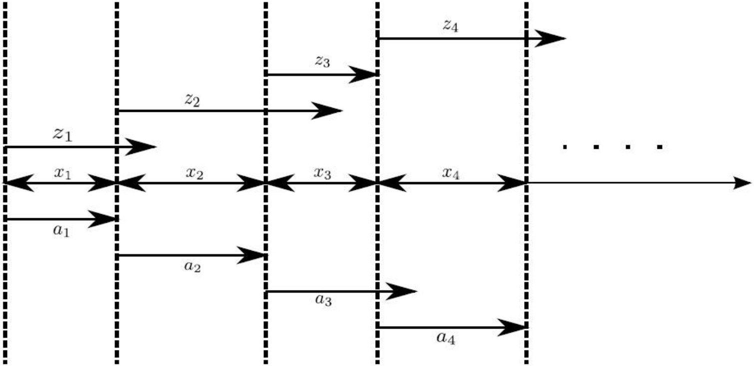


Figure: $\alpha = 1, \beta = 4.5, C_c = 10C_p$

Second policy: Dynamic policy

$\longrightarrow$ The duration **a** is adaptative and depends on the past of maintenance process.



Dynamic policy (II)

$\longrightarrow$ At the beginning, the initial age follows the limiting distribution A_∞.

$\longrightarrow$ Given the initial age u, the age after the ith maintenance is

$$A_i(u) = (1-\rho)^i u + \sum_{j=1}^{i} (1-\rho)^{i-j+1} x_j$$

$\longrightarrow$ The distribution of the next inter-failure time can be computed.

$$R_{Z_{i+1}}(z) = -\int_0^\infty \frac{e^{-\alpha(A_i(u)+z)^\beta}}{e^{-\alpha A_i(u)^\beta}} dR_{A_\infty}(u)$$

$\longrightarrow$ The optimal PM should be carried out after a duration a_{i+1}^*.

$$a_{i+1}^* = arg \min_a \frac{C_p + (C_c - C_p)(1 - R_{Z_{i+1}}(a))}{\int_0^a R_{Z_{i+1}}(u) du}$$

Third policy: Failure limit policy

$\longrightarrow$ After a maintenance (corrective or preventive) has restored the system, a new PM is scheduled when the virtual age of the system exceeds a threshold $\mathcal{A}$.

$\longrightarrow$ "Virtual age limit" policy.

Comparing the costs

Table: Optimal maintenance strategies ($\alpha = 1$, $C_c = 10C_p$)

β	ρ	Static	age	Failure limit	Variant I	age	Dynamic	no PM
1.5	0.2	17.72	0.18	17.69	20.81	0.84	20.76	22.01
1.5	0.5	12.30	0.26	12.28	13.51	0.62	13.11	15.68
1.5	0.8	9.71	0.33	9.71	9.97	0.48	9.72	12.74
3	0.2	15.50	0.09	15.48	25.00	0.23	23.64	40.70
3	0.5	7.54	0.20	7.54	8.02	0.26	7.62	20.72
3	0.8	4.92	0.30	4.92	4.92	0.30	4.92	13.90
4.5	0.2	12.51	0.10	12.51	16.51	0.15	16.10	46.87
4.5	0.5	5.49	0.23	5.48	5.51	0.22	5.49	21.39
4.5	0.8	3.46	0.37	3.46	3.49	0.34	3.46	13.68

Analysis

- The failure limit (virtual age limit) policy (III) is the most efficient.
- The age-dependent policy (I) is almost as powerful.
- In practice, the age-dependent policy seems simpler to implement than the failure limit policy.
- The dynamic policy (II) is locally optimal, but is outperformed by the previous static policies.
- The approximation of the age-dependent policy (I') offers decent results.
- As this approximation does not take into account the dependency between consecutive inter-failure times, its validity is poor with small ρ.

Conclusion

Future work

- Add estimation procedures within the optimal PM strategy.
- Dissociate the maintenance efficiencies.
- Take into account the downtime costs.
- Application to real dataset
- Develop goodness of fit procedures for the WARA_∞ model.